Problem Sheet 2, v2

1. i) Draw the graphs for $[x]$ and $\{x\}$.

   ii) Show that for $\alpha \in \mathbb{R}$,

$$\int_{\alpha}^{\alpha+1} [t]\, dt = \alpha \qquad \text{and} \qquad \int_{\alpha}^{\alpha+1} \{t\}\, dt = \frac{1}{2}.$$

   Hint Split these integrals at the integer which must lie in any interval of length 1, such as $[\alpha, \alpha+1]$.

   iii) Prove that for $x > 1$

$$\begin{aligned} \int_{1}^{x} [t]\, dt &= \frac{1}{2}[x]([x]-1) + \{x\}[x] \\ &= \frac{1}{2}x(x-1) + \frac{1}{2}\{x\}(1-\{x\}). \end{aligned}$$

   This is often written as

$$\int_{1}^{x} [t]\, dt = \frac{1}{2}x(x-1) + O(1),$$

   though the error term is zero when $x \in \mathbb{Z}$. The result will be used in the course in the form

$$\int_{1}^{x} [t]\, dt = \frac{1}{2}x^2 + O(x),$$

   (where the error is not now zero when x is an integer.)

2. *How fast does the logarithm grow?*.

   i) Recall the fundamental idea from Chapter 1,

$$\left(\inf_{t\in[y,x]} f(t)\right)(x-y) \le \int_{y}^{x} f(t)\, dt \le \left(\sup_{t\in[y,x]} f(t)\right)(x-y). \tag{27}$$

   Use (27) to show that, for all real $y > 1$,

$$\log y < y - 1 < y.$$

ii) By an appropriate choice of y in part i show that for all $n \geq 1$ we have

$$\log x < nx^{1/n} \tag{28}$$

for $x > 1$.

iii) Deduce that for all $\varepsilon > 0$, we have $\log x \ll_\varepsilon x^\varepsilon$. This means that **the logarithm of x grows slower than any power of x.**

Here $\log x \ll_\varepsilon x^\varepsilon$ means there exists a constant $C(\varepsilon)$, depending on ε, for which $\log x \leq C(\varepsilon)\, x^\varepsilon$ for all $x \geq 1$.

iv) In the notes we make use of both of

$$x^{1/3} \log x < Cx^{1/2} \quad \text{and} \quad x^\delta < C\frac{x}{\log^\ell x},$$

for any constant $C > 0$, $\delta < 1$ and $\ell \geq 1$. Prove these inequalities both hold for sufficiently large x.

3. *A technical result used in more advanced results on the Prime Number Theorem.*

For a function whose growth is

- **faster** than x^δ for any $\delta < 1$ yet
- **slower** than $x/\log^\ell x$ for any $\ell \geq 1$,

consider

$$x \exp\left(-C\left(\log x\right)^\alpha\right)$$

with constants $C > 0$ and $\alpha > 0$.

Prove that

a) If $\delta < 1$ then for any $C > 0$

$$x^\delta \leq x \exp\left(-C\left(\log x\right)^\alpha\right)$$

for all sufficiently large x as long as $\alpha < 1$.

b) If $\ell \geq 1$ then for any $C > 0$

$$x \exp\left(-C\left(\log x\right)^\alpha\right) \leq \frac{x}{\log^\ell x}$$

for all sufficiently large x as long as $\alpha > 0$.

4. i) *Estimates of integrals found in error terms.* Show that for $\alpha \geq 0$ and $\ell \geq 1$ we have.

$$\int_2^x \frac{t^\alpha}{\log^\ell t} dt \ll_{\ell,\alpha} \frac{x^{1+\alpha}}{\log^\ell x}.$$

Hint Split the integral at $\sqrt{x}$ and use the fact that $\log t$ is an increasing function of t.

ii) Show that for $\alpha > 1$ and $\ell \geq 1$ we have

$$\int_x^\infty \frac{\log^\ell t}{t^\alpha} dt \ll_{\ell,\alpha} \frac{\log^\ell x}{x^{\alpha-1}}.$$

Hint Split the integral at x^2 and use (27). In the shorter interval again use that $\log t$ is an increasing function while in the longer interval use $\log t \ll t^\varepsilon$ with some appropriately chosen ε.

iii) *A result used in more advanced results on the Prime Number Theorem.* Show that with $\alpha > 0$ and $C > 0$ we have

$$\int_2^x \exp\left(-C\left(\log t\right)^\alpha\right) dt \ll x \exp\left(-C'\left(\log x\right)^\alpha\right),$$

for some $C' > 0$.

Problem Sheet 2: Generalising Euler's constant.

Recall from lectures that a result for comparing sums with integrals is that if f has a continuous derivative, is non-negative and monotonic then

$$\sum_{1\leq n\leq x} f(n) = \int_1^x f(t)\,dt + O(\max\left(f(1), f(x)\right)), \tag{29}$$

for all real $x \geq 1$.

In fact we deduced (29) from the **Euler Summation**: Let f have a continuous derivative $x > 0$. Then

$$\sum_{1\leq n\leq x} f(n) = \int_1^x f(t)\,dt + f(1) - \{x\}\,f(x) + \int_1^x \{t\}\,f'(t)\,dt$$

for all real $x \geq 1$.

As an application of (29) we showed that there exists a constant γ such that

$$\sum_{1\leq n\leq x} \frac{1}{n} = \log x + \gamma + O\left(\frac{1}{x}\right), \tag{30}$$

for $x \geq 1$. The essential idea here is that we come across a convergent integral

$$\int_1^x \frac{\{t\}}{t^2}dt.$$

This could simply be bounded as $O(1)$, but for a better result it is *completed to infinity* and the tail end, the integral from x to ∞, is *bounded* (often using the results of Question 4). This is a standard method and is used in the following question in which we generalise (30) to a sum of $(\log n)^\ell/n$ for any integer $\ell \geq 1$. The result of Question 5 is used later when examining the Laurent Expansion of $\zeta(s)$.

5. Prove that for all $\ell \geq 1$ there exists a constant C_ℓ such that

$$\sum_{n\leq x} \frac{\log^\ell n}{n} = \frac{1}{\ell+1}\log^{\ell+1} x + C_\ell + O\left(\frac{\log^\ell x}{x}\right),$$

for all real $x \geq 1$.

The result when $\ell = 0$ has $C_0 = \gamma$, Euler's constant. Notice how we have the best possible error term.

Problem Sheet 2: More Sums of logs.

6. *Generalise a result in lectures written (in a weakened form) as*

$$\sum_{1\le n\le x} \log n = x\log x + O(x)$$

i) With integer $\ell \ge 1$ justify

$$\sum_{1\le n\le x} \log^{\ell} n = \int_1^x \log^{\ell} t dt + O\big(\log^{\ell} x\big) \tag{31}$$

for all real $x \ge 1$.

ii) Prove that

$$\int_1^x \log^{\ell} t dt = x\log^{\ell} x + O_{\ell}\big(x\log^{\ell-1} x\big)\,.$$

Deduce

$$\sum_{1\le n\le x} \log^{\ell} n = x\log^{\ell} x + O_{\ell}\big(x\log^{\ell-1} x\big)\,.$$

for all real $x \ge 1$.

Note the best error term here would be $O\big(\log^{\ell} x\big)$, far smaller than the one here.

7. *Improve the result of Qu 6 to the best possible error term.*

Change the variable of integration in (31) to $u = \log t$ so

$$\int_1^x \log^{\ell} t dt = \int_0^{\log x} e^u u^{\ell} du$$

i) Prove by induction that

$$\int_0^y e^u u^d du = e^y \sum_{r=0}^{d} (-1)^r \frac{d!}{(d-r)!} y^{d-r} - (-1)^d d! \tag{32}$$

for all $d \ge 0$.

ii) Prove that for any integer $\ell \geq 0$ we have

$$\sum_{n \leq x} \log^{\ell} n = x P_{\ell}(\log x) + O\left(\log^{\ell} x\right),$$

where

$$P_d(y) = \sum_{r=0}^{d} (-1)^r \frac{d!}{(d-r)!} y^{d-r},$$

a polynomial of degree d.

Problem Sheet 2: Partial Summation

Partial Summation is often no more than an exercise in interchanging Sums and Integrals.

8. Let $x \geq 1$ be real, $\{a_n\}_{n \geq 1}$ a sequence of complex numbers and $A(x) = \sum_{1 \leq n \leq x} a_n$.

a) Show that

$$\sum_{1 \leq n \leq x} a_n (x - n) = \int_1^x A(t)\, dt.$$

b) Show that

$$\sum_{1 \leq n \leq x} a_n \log \frac{x}{n} = \int_1^x \frac{A(t)}{t} dt = \int_0^{\log x} A(e^y)\, dy.$$

c) Show that

$$\sum_{1 \leq n \leq x} a_n (e^x - e^n) = \int_1^x e^t A(t)\, dt = \int_e^{e^x} A(\log y)\, dy.$$

Hint Write $x - n$, $\log(x/n)$ and $e^x - e^n$ as integrals.

9. *The last question concerned sums over $n \leq x$, this question will be for sums over $n > x$.*

For f with a continuous derivative for $x > 0$ satisfying $f(x) \to 0$ as $x \to \infty$ and $\int_1^{\infty} |f'(t)|\, dt < \infty$, then

$$\sum_{n>x} a_n f(n) = -A(x)\, f(x) - \int_x^\infty A(t)\, f'(t)\, dt.$$

for sequences $\{a_n\}$ for which the sum and integral converge.

Hint Write $f(n)$ as an integral from n to ∞.

10. a) Use Partial Summation to prove that if f has a *continuous derivative* on $[1, x]$ then for a sum *over primes* we have

$$\sum_{p \le x} f(p) = -\int_2^x \pi(t)\, f'(t)\, dt + \pi(x)\, f(x)\,. \tag{33}$$

b) Recalling that $\theta(x) = \sum_{p \le x} \log p$, deduce that

$$\theta(x) = \pi(x) \log x + O\left(\frac{x}{\log x}\right).$$

Compare this with Theorem 2.20.

Hint for b**.** You may have to use Chebyshev's bound $\pi(x) = O(x/\log x)$ in the integral that arises along with Question 4

Problem Sheet 2: Deductions from Merten's Theorem.

11. Prove that

$$\sum_{2 \le n \le x} \frac{1}{n \log n} = \log\log x + O(1)\,.$$

12. On the previous Problem Sheet you were asked to show that

$$\sum_{n=2}^{\infty} \frac{1}{n\,(\log n)^{1+\alpha}} \tag{34}$$

converges for all $\alpha > 0$.

Question 11 could be compared with Merten's result

$$\sum_{p \le x} \frac{1}{p} = \log\log x + O(1)\,.$$

This may lead you to think that the n-th prime is "of size" $n \log n$, which we might write as $p_n \approx n \log n$. In which case $\log p_n$ may be thought of as of size

$$\log p_n \approx \log(n \log n) = \log n + \log\log n \approx \log n.$$

Thus $p_n(\log p_n)^\alpha$ would be "approximately" of the size

$$p_n(\log p_n)^\alpha \approx (n \log n)(\log n)^\alpha = n(\log n)^{1+\alpha}.$$

Hence the convergence of (34) might then suggest that the sum over primes

$$\sum_p \frac{1}{p\,(\log p)^\alpha}.$$

converges for all $\alpha > 0$. Prove that this is so.

Hint Use Partial Summation to remove the $1/\left(\log p\right)^\alpha$ so you can apply Merten's Theorem, Theorem 2.22, and in particular (18).

13. Prove that for a fixed $c > 1$,

$$\sum_{x<n\le cx} \frac{\Lambda(n)}{n} \ll 1,$$

i.e. this sum is bounded for all $x > 1$.

(What is more difficult, and thus of more interest, is to show that this sum is bounded **below** and thus non-zero. For this would then show the existence of $n \in [x, cx]$ for which $\Lambda(n) \neq 0$. This n would be a power of a prime, and since powers greater or equal to 2 are rare, this would lead to the existence of a prime in $[x, cx]$ for any $c > 1$.)

Hint Use Merten's result (16) (*twice*).

14. Prove that

$$\int_1^x \frac{\psi(u)}{u^2} du = \log x + O(1)\,.$$

Hint Interchange the integral and the summation within the definition of ψ, use Merten's Theorem and Chebyshev's bound $\psi(x) \ll x$.

15. Prove that for a fixed constant $c > 1$,

$$\int_1^x \frac{\psi(cu) - \psi(u)}{u^2} du = (c-1) \log x + O_c(1) \, .$$

Note, Since for sufficiently large x the right hand side is greater than 0 we must have that the integrand is non-zero, thus $\psi(cu) - \psi(u) > 0$, and in particular, there is a prime in $[u, cu]$, for *some* values of u. Unfortunately this does not tell us for which u these intervals contain a prime (it is, in fact, for all u sufficiently large) and how many primes are in these intervals.

Problem Sheet 2: Prime Number Theorem

16. (Tricky) Assume the Prime Number Theorem in the form $\pi(x) \sim x/\log x$ as $x \to \infty$. Prove that the n-th prime p_n satisfies

$$p_n \sim n \log n$$

as $n \to \infty$, i.e.

$$\lim_{n\to\infty} \frac{p_n}{n \log n} = 1.$$

Hint Note that $\pi(p_n) = n$, apply the Prime Number Theorem in the form given in the question and take logarithms.

The question justifies the assumption in Question 12.